ISRAEL JOURNAL OF MATHEMATICS, Vol. 28, Nos. 1-2, 1977

BERNOULLI MAPS OF THE INTERVAL

BY

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ABSTRACT

The ergodic properties of expanding piecewise C^2 maps of the interval are studied. It is shown that such a map is Bernoulli if it is weak-mixing. Conditions are given that imply weak-mixing (and hence Bernoulliness).

Suppose that f: $[0,1] \to [0,1]$ is a piecewise C^2 function, i.e., there is a finite partition $0 = a_0 < a_1 < \cdots < a_r = 1$ so that each $f \mid (a_i, a_{i+1})$ extends to a C^2 function on $[a_i, a_{i+1}]$. If $\lambda = \inf_{0 \leqq x \leqq 1} |f'(x)| > 1$, then Lasota and Yorke [6] showed that f possesses a smooth invariant probability measure μ. This means that $\mu(E) = \int_E p(x)\,dx$ where $p \in L_1$ and that $\mu(f^{-1}E) = \mu(E)$ for all Borel sets E. In this paper we study the ergodic properties of such a μ.

THEOREM 1. *Let f be a piecewise C^2 map of $[0,1]$, $\lambda = \inf_{0 \leqq x \leqq 1} |f'(x)| > 1$, and μ be a smooth f-invariant probability measure. If (f, μ) is weak-mixing, then the natural extension of (f, μ) is Bernoulli.*

THEOREM 2. *With f and μ as before, (f, μ) will be weak-mixing if one of the following holds:*

(a) $\sup_{n>0} \mu(f^n U) = 1$ *for all nonempty open intervals U with $\mu(U) > 0$,*

(b) $r = 2$ *and* $\lambda > \sqrt{2}$,

(c) $\lambda > 2$ *and condition* (a) *holds for the sets* $U = (a_j, a_{j+1})$, $1 \leqq j \leqq r-2$.

These results are along the lines of numerous previous papers on maps of the interval [1, 6, 7, 9, 11, 12, 15, 18, 19]. The situation is analogous to that for Anosov-like examples in dynamical systems [2, 3, 5, 8, 10, 13, 14] in that a topological condition implies weak-mixing and this in turn yields Bernoulliness. Our proofs in fact are mostly just translations of these papers in dynamical systems into the language of mappings of the interval.

PRELIMINARIES. If $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of $[0,1]$, we say $\mathcal{P}$ and $\mathcal{Q}$ are *ε-independent* if

[†] Partially supported by NSF Grant MCS74-19388 and the Sloan Foundation.
Received September 21, 1976

$$D(\mathscr{P},\mathscr{Q})=\sum_{P\in\mathscr{P},Q\in\mathscr{Q}}|\mu(P\cap Q)-\mu(P)\mu(Q)|<\varepsilon.$$

One defines new partitions $\mathscr{P}\vee\mathscr{Q}=\{P\cap Q:\ P\in\mathscr{P},\ Q\in\mathscr{Q}\}$, $f^{-n}\mathscr{P}=\{f^{-n}P:\ P\in\mathscr{P}\}$, and $\bigvee_{n=m}^{M}f^{-n}\mathscr{P}=f^{-m}\mathscr{P}\vee\cdots\vee f^{-M}\mathscr{P}$. A partition $\mathscr{P}$ is called *weak-*Bernoulli if $\forall\varepsilon>0\ \exists N$ so that $D(\bigvee_{n=0}^{k}f^{-n}\mathscr{P},\bigvee_{n=k+N}^{2k+N}f^{-n}\mathscr{P})<\varepsilon$ for all $k\geqq 0$. It is easy to show that it suffices to check this condition for all sufficiently large k. Through this paper $\mathscr{P}$ will always denote the partition $\mathscr{P}=\{(0,a_1),(a_1,a_2),\cdots,(a_{r-1},1)\}$. Theorem 1 is proved by showing $\mathscr{P}$ is a weak-Bernoulli partition [4].

One sees inductively that $\bigvee_{n=0}^{k}f^{-n}\mathscr{P}$ is a partition into intervals of length $\leqq\lambda^{-k}$. The condition $\lambda=\inf_x|f'(x)|>1$ in particular implies that each $f\,|\,[a_i,a_{i+1}]$ is strictly monotonic.

Lemma 1. *If $A\in\bigvee_{n=0}^{k}f^{-n}\mathscr{P}$, $A\neq\varnothing$, and $\bar{A}\cap\{a_0,a_1,\cdots,a_r\}=\varnothing$, then $fA\in\bigvee_{n=0}^{k-1}f^{-n}\mathscr{P}$.*

Proof. Let $A=\bigcap_{n=0}^{k}f^{-n}(a_{i_n},a_{i_n+1})$. Then $A=(a_{i_0},a_{i_0+1})\cap f^{-1}B$ where $B=\bigcap_{n=1}^{k}f^{-n+1}(a_{i_n},a_{i_n+1})$. To show $fA=B$ it is enough to see that $f(a_{i_0},a_{i_0+1})\supset B$. Now f maps (a_{i_0},a_{i_0+1}) monotonically onto an open interval which intersects B (as $A\neq\varnothing$). Unless $f(a_{i_0},a_{i_0+1})\supset B$ we must have $\bar{B}\cap\{f(a_{i_0}),f(a_{i_0+1})\}\neq\varnothing$ and also $\overline{B\cap f(a_{i_0},a_{i_0+1})}\cap\{f(a_{i_0}),f(a_{i_0+1})\}\neq\varnothing$. It follows that either a_{i_0} or a_{i_0+1} is in $\bar{A}$. □

Lemma 2. *Given $\alpha>0$, for N sufficiently large, there is a collection of atoms $\mathscr{A}_N\subset\bigvee_{n=0}^{N}f^{-n}\mathscr{P}$ so that $\mu(\cup\mathscr{A}_N)>1-\alpha$ and $p(x)/p(y)\in[e^{-\alpha},e^{\alpha}]$ for $x,y\in A\in\mathscr{A}_N$.*

Proof. This proof depends on the fact that $p(x)$ is a function of bounded variation [6]. Consider the following exhaustive list of possibilities for an atom $A\in\bigvee_{n=0}^{N}f^{-n}\mathscr{P}$.

(i) $p(x)\geqq\alpha/2$ for all $x\in A$ and $p(y)>e^{\alpha}p(z)$ for some $y,z\in A$.

(ii) $p(x)\leqq\alpha/2$ and $p(y)\geqq 3\alpha/4$ for some $x,y\in A$.

(iii) $p(x)\leqq 3\alpha/4$ for all $x\in A$.

(iv) $p(x)\geqq\alpha/2$ for all $x\in A$ and $p(y)\leqq e^{\alpha}p(z)$ for all $y,z\in A$.

Let $K=\|p\|_\infty+$ total variation of $p(x)$ on $[0,1]$. The variation of $p(x)$ over an A satisfying (i) or (ii) is at least $\gamma=\min\{(e^{\alpha}-1)\alpha/2,\alpha/4\}$. The total number of such atoms A is at most $K\gamma^{-1}$ and the total μ-measure of such atoms at most $K^2\gamma^{-1}\lambda^{-N}$. The total μ-measure of all atoms satisfying (iii) is at most $3\alpha/4$. For N large one has that the measure of the atoms satisfying (iv) is at least $1-\alpha$. □

Lemma 3. *Given $\beta>0$, there is an $M=M(\beta)$ so that for each $m\geqq 0$ one can find a collection of atoms $\mathscr{B}=\mathscr{B}_{m+M}\subset\bigvee_{n=0}^{m+M}f^{-n}\mathscr{P}$ with*

(i) $f^m B \in \vee_{m=0}^{M} f^{-n}\mathcal{P}$ *for* $B \in \mathcal{B}$,

(ii) $\dfrac{\mu(\tilde{B})}{\mu(B)} \in \dfrac{\mu(f^m\tilde{B})}{\mu(f^m B)}[e^{-\beta}, e^{\beta}]$ *for* $\tilde{B} \subset B$, $B \in \mathcal{B}$, *and*

(iii) $\mu(\cup \mathcal{B}) > 1 - \beta$.

Proof. By Lemma 1 and induction on m, (i) will hold for B unless at least one of the sets $\bar{\tilde{B}}, \overline{fB}, \cdots, \overline{f^{m-1}B}$ intersects $\{a_0, \cdots, a_r\}$. Now $f^k B$ lies in an atom of $\vee_{n=0}^{M+m-k} f^{-n}\mathcal{P}$; as these atoms have measure at most $K\lambda^{-(M+m-k)}$ and at most $2r$ of them are adjacent to an a_i, the total μ-measure of all B's in $\vee_{n=0}^{m+M} f^{-n}\mathcal{P}$ with $\overline{f^k B} \cap \{a_0, \cdots, a_r\} \neq \emptyset$ is at most $2rK\lambda^{-(M+m-k)}$ (using that μ is f-invariant). Thus the B's for which (i) holds have total measure at least

$$1 - \sum_{k=0}^{m} 2rK\lambda^{-M-m+k} \geqq 1 - \left(\frac{2rK\lambda^{-M}}{1-\lambda^{-1}}\right).$$

This is greater than $1 - \frac{1}{3}\beta$ for M large.

For $\tilde{B} \subset B$ the change of variables formula gives us ($f^m \mid B$ is one-to-one)

$$\mu(f^m\tilde{B}) = \int_{f^m\tilde{B}} p(y)\,dy = \int_{\tilde{B}} p(f^m x)\,|(f^m)'(x)|\,dx$$
$$= \int_{\tilde{B}} \left\{\frac{p(f^m x)\,|(f^m)'(x)|}{p(x)}\right\} p(x)\,dx.$$

Since f is piecewise C^2 and $\lambda = \inf|f'(x)| > 1$, we can find a constant d so that

$$\left|\frac{f'(u)}{f'(v)}\right| \in [e^{-d|u-v|}, e^{d|u-v|}] \quad \text{when} \quad u, v \in [a_i, a_{i+1}].$$

Then for $u, v \in B \in \vee_{n=0}^{m+M} f^{-n}\mathcal{P}$ we have $|f^k u - f^k v| \leqq \lambda^{-(m+M-k)}$ and

$$\left|\frac{(f^m)'(u)}{(f^m)'(v)}\right| = \prod_{k=0}^{m-1}\left|\frac{f'(f^k u)}{f'(f^k v)}\right| \in [e^{-d^*\lambda^{-M}}, e^{d^*\lambda^{-M}}]$$

where $d^*\lambda^{-M} = d\sum_{j=0}^{\infty}\lambda^{-M-j} = d\lambda^{-M}/(1-\lambda^{-1})$. On the other hand, by Lemma 2, for M large the functions $p(x)$ and $p(f^m x)$ will each vary by at most a multiplicative factor in $[e^{-\beta/3}, e^{\beta/3}]$ as x runs over B, except when either B or $f^m B$ is in certain sets of atoms, each of total measure $\beta/3$. Except therefore for a set $\mathcal{B}^c$ of atoms B with total measure less than β, one will have that (i) holds and that the function $p(f^m x)|(f^m)'(x)|/p(x)$ differs from a constant by a factor in $[e^{-\beta}, e^{\beta}]$. This implies (ii). □

Proof of Theorem 2. One calls the endomorphism (f, μ) weak-mixing provided its natural extension $(\tilde{f}, \tilde{\mu})$ is [12]. We claim this is equivalent to the following condition: whenever F is a bounded measurable function on $[0,1]$ and $\tau \in C$ satisfies

$$F(fx) = \tau F(x) \quad \text{for} \quad \mu\text{-a.e. } x,$$

then F is μ-equivalent to a constant. Note that $|\tau| = 1$ because f preserves μ. It is well known that this nonexistence of nonconstant eigenfunctions is equivalent to weak-mixing in the invertible case. Now the operators U_f on $L_2(\mu)$ defined by $U_f F(x) = F(fx)$ and $U_{\tilde f}$ on $L_2(\tilde\mu)$ are related as follows [12]: there is a closed subspace $H \subset L_2(\tilde\mu)$ so that $U_{\tilde f}(H) \subset H$, $U_{\tilde f} \mid H$ is isomorphic to U_f on $L_2(\mu)$ and $L_2(\tilde\mu) = \overline{\bigcup_{n=0}^{\infty} U_{\tilde f}^{-n} H}$. If $F \in L_2(\mu)$ satisfies $U_f F = \tau F$, this gives an $\tilde F \in H$ with $U_{\tilde f}\tilde F = \tau \tilde F$. On the other hand, suppose $\tilde F \in L_2(\tilde\mu)$ satisfies $U_{\tilde f}\tilde F = \tau\tilde F$. For each $\varepsilon > 0$, choose $G \in U_{\tilde f}^{-n} H$, some n, with $\|G - \tilde F\|_2 < \varepsilon$. Then $\tilde F = U^n(\tilde F - G)/\tau^n + U^n G/\tau^n$, $\|\tilde F - U^n G/\tau^n\|_2 < \varepsilon$ and $U^n G/\tau^n \in H$. Letting $\varepsilon \to 0$, $\tilde F \in H$ and so there is an $F \in L_2(\mu)$ with $U_f F = F$. Thus one sees that weak-mixing for (f, μ) is equivalent to the nonexistence of nonconstant eigenfunctions F for U_f, as claimed above.

Choose $M = M(\beta)$ as in Lemma 3 for $\beta > 0$. Then for each $m \geqq 0$ let $\mathscr{C}$ be the collection of atoms $C \in \bigvee_{n=0}^{M} f^{-n}\mathscr{P}$ so that at least $1 - \sqrt{\beta}$ (in terms of μ-measure) of the atoms $B \in \bigvee_{n=0}^{m+M} f^{-n}\mathscr{P}$ with $f^m B \subset C$ satisfy $B \in \mathscr{B}_{m+M}$. Then $\mu(\cup \mathscr{C}_m) \geqq 1 - \sqrt{\beta}$. One can pick C with $\mu(C) > 0$ and $m_k \to \infty$ so that $C \in \mathscr{C}_{m_k}$ for all k.

For any $\delta > 0$ there is a compact set K with $F \mid K$ continuous and $\mu(K) > 1 - (1 - \sqrt{\beta})\delta\mu(C)$. For at least one atom $B \in \mathscr{B}_{m_k+M}$ with $f^{m_k}B = C$ one must have $\mu(B \cap K^c) \leqq \delta\mu(B)$. Choose k large enough so that

$$x, y \in K, \qquad |x - y| < \lambda^{-m_k} \Rightarrow |F(x) - F(y)| < \delta.$$

Then F varies by at most δ on $B \cap K$, and so F varies by at most δ on $f^{m_k}(B \cap K)$ (as $|\tau| = 1$). Now $f^{m_k}B = C$ and (by Lemma 3)

$$\mu(f^{m_k}(B \cap K^c)) \leqq \frac{\mu(C)\mu(B \cap K^c)}{\mu(B)} e^{\beta} \leqq \delta e^{\beta}\mu(C).$$

Thus F varies by at most δ on a subset of C of measure $\geqq (1 - \delta e^{\beta})\mu(C)$. Letting $\delta \to 0$ we have that F is constant μ-a.e. on C.

The above argument worked for all C in the family

$$\mathscr{C}(\beta) = \{C \in \bigvee_{n=0}^{M} f^{-n}\mathscr{P} : C \in \mathscr{C}_m \text{ for infinitely many } m > 0\}.$$

Now $\mu(\mathscr{C}(\beta)) \geqq 1 - \sqrt{\beta}$ since $\mu(\mathscr{C}_m) > 1 - \sqrt{\beta}$. Letting $\beta \to 0$ we see that there is a countable collection $\mathcal{I}_F = \{I_1, I_2, \cdots\}$ of disjoint open intervals so that F is μ-equivalent to a constant on each I_j, each I_j is an atom of some $\bigvee_{n=0}^{M_j} f^{-n}\mathcal{P}$, and $\mu(\cup I_j) = 1$. Now F is μ-equivalent to a constant on $f^n I_j$; because of the equation $F(f^n x) = \tau^n F(x)$ μ-a.e. x. Part (a) of Theorem 2 is now clear.

Now $\tau^n = 1$ for some $n > 0$; otherwise $I_j, f^{-1}I_j, f^{-2}I_j, \cdots$ would be μ-disjoint sets with equal positive measure. By lemma 2.8 of [7] there is a set $J_1, \cdots, J_m$ of disjoint closed intervals so that $p(x) > 0$ for Lebesgue a.e. $x \in \bigcup_{i=1}^m J_i$ and $p(x) = 0$ for Lebesgue a.e. $x \notin \bigcup_{i=1}^m J_i$. Call an open interval U *μ-positive* if $p(x) > 0$ for Lebesgue a.e. $x \in U$. For $c \in R$ call U a *maximal c-interval* if

(i) U is a μ-positive open interval,

(ii) $F(x) = c$ for a.e. $x \in U$, and

(iii) if $V \supset U$ satisfies (i) and (ii), then $V = U$.

Because of the properties of the I_j's and J_i's, for each $c \in R$ the set $\{x \in [0,1]: F(x) = c\}$ is μ-equivalent to the union of the (countable) family of maximal c-intervals.

Let c be the essentially constant value of F on I_1, and let $U_1, U_2, \cdots$ be the maximal c-intervals. We may assume that length $U_1 \geqq$ length U_k for all k.

Now assume condition (c) holds. If U is a μ-positive open interval then either

(i) $U \supset (a_i, a_{i+1})$ for some $i \in [1, r-2]$, or

(ii) $f(U)$ contains a μ-positive open interval V with length $V >$ length U.

For if (i) does not hold either $f \mid U$ is monotonic or $U = \tilde{U}_1 \cup \tilde{U}_2$ with $f \mid \tilde{U}_1$, $f \mid \tilde{U}_2$ both monotonic; one of $\tilde{U}_1$, $\tilde{U}_2$ is longer and f expands lengths by at least $\lambda > 2$; $f(U)$ is μ-positive because μ is f-invariant. Define intervals $W_0, W_1, \cdots, W_n$ inductively by $W_0 = U_1$; $W_{j+1} = V$ when W_j satisfies (ii). If any W_j satisfied (i) (so the definition of W_n did not work), then $\sup_{n>0} \mu(f^n U_1) = 1$ and F is equivalent to a constant. Now $F(W_n) = c$ since $\tau^n = 1$ and W_n is a μ-positive open interval with bigger length than $W_0 = U_1$. This is a contradiction — so in fact some W_j satisfied (i) above.

Assume now condition (b), i.e., $r = 2$ and $\lambda > \sqrt{2}$. Now Li and Yorke [7] have shown in the case $r = 2$ that μ is ergodic for f. For $n = 1$ this gives F constant. Now define the intervals $W_0, \cdots, W_n$ be letting $W_0 = U_1$ and $W_{j+1} =$ longest μ-positive open interval contained in fW_j. As $F(W_j) = c\tau^j$, these W_j's are disjoint. So at most one index j_0 has a_1 W_{j_0} . Here one has length $W_{j_0+1} \geqq \lambda/2$ length W_{j_0}. For all other j one has length $W_{j+1} \geqq \lambda$ length W_j. So length $W_n \geqq \lambda^n/2$ length W_0 which gives a contradiction for $n \geqq 2$. □

Proof of Theorem 1. If $\tilde{\mathcal{Q}}, \tilde{\mathcal{R}}$ are collections of atoms of partitions $\mathcal{Q}, \mathcal{R}$, then

$$D(\mathcal{Q}, \mathcal{R}) \leqq 2(2 - \mu(\cup \tilde{\mathcal{Q}}) - \mu(\cup \tilde{\mathcal{R}})) + D(\tilde{\mathcal{Q}}, \tilde{\mathcal{R}}).$$

Let $\beta > 0$ and choose M as in Lemma 3. N is a positive integer to be determined later. We will estimate

$$\gamma_{N,m} = D\left(\bigvee_{n=0}^{m+M} f^{-n}\mathcal{P}, \bigvee_{n=m+M+N}^{2m+2M+N} f^{-n}\mathcal{P}\right)$$

$$\leqq 2\beta + D\left(\mathcal{B}_{m+M}, \bigvee_{n=m+M+N}^{2m+2M+N} f^{-n}\mathcal{P}\right).$$

For $B \in \mathcal{B}_{m+M}$ and $D \in \bigvee_{n=m+M+N}^{2m+2M+N} f^{-n}\mathcal{P}$, one has

$$\frac{\mu(B \cap D)}{\mu(B)} \in \frac{\mu(f^m B \cap f^m D)}{\mu(f^m B)} [e^{-\beta}, e^{\beta}].$$

Using this and $\mu(f^m D) = \mu(D)$ (as D is $\bigvee_{n=m}^{\infty} f^{-n}\mathcal{P}$ measurable), we see

$$D(\mathcal{B}_{m+M}, \bigvee_{n=M+m+N}^{2M+2m+N} f^{-n}\mathcal{P}) = \sum_B \sum_D |\mu(B \cap D) - \mu(B)\mu(D)|$$

$$\leqq \sum_B \mu(B) \sum_D \left\{ \left| \frac{\mu(f^m B \cap f^m D)}{\mu(f^m B)} - \mu(f^m D) \right| + (e^{\beta} - 1)\frac{\mu(f^m B \cap f^m D)}{\mu(f^m B)} \right\}.$$

$$\leqq (e^{\beta} - 1) + D\left(\bigvee_{n=0}^{M} f^{-n}\mathcal{P}, \bigvee_{n=M+N}^{2M+m+N} f^{-n}\mathcal{P}\right).$$

Next we consider atoms $\mathcal{B}_{2M+N} \subset \bigvee_{n=0}^{2M+N} f^{-n}\mathcal{P}$ of Lemma 3. Let $\mathcal{E}_N$ be the collection of atoms $E \in \bigvee_{n=0}^{M} f^{-n}\mathcal{P}$ such that $\tilde{E}_N = E \cap \bigcup \mathcal{B}_{2M+N}$ satisfies $\mu(\tilde{E}_N) \geqq (1 - \sqrt{\beta})\mu(E)$. Then $\mu(\bigcup \mathcal{E}_N) > 1 - \sqrt{\beta}$. We now have

$$\gamma_{N,m} \leqq (2\beta + e^{\beta} - 1) + 2\sqrt{\beta} + D(\mathcal{E}_N, \bigvee_{n=M+N}^{2M+N+m} f^{-n}\mathcal{P}).$$

Since, for $E \in \mathcal{E}_N$ and F running over $\bigvee_{n=M+N}^{2M+N+m} f^{-n}\mathcal{P}$, one has

$$\sum_F |\mu(E \cap F) - \mu(\tilde{E}_N \cap F)| \leqq \mu(E \backslash \tilde{E}_N) \leqq \sqrt{\beta}\mu(E)$$

and

$$\sum_F |\mu(\tilde{E}_N)\mu(F) - \mu(E)\mu(F)| \leqq |\mu(\tilde{E}_N) - \mu(E_N)| \leqq \sqrt{\beta}\mu(E),$$

one sees

$$\gamma_{N,m} \leqq (2\beta + e^{\beta} - 1 + 4\sqrt{\beta}) + D(\{\tilde{E}_N : E \in \mathcal{E}_N\}, \bigvee_{n=M+N}^{2M+N+m} f^{-n}\mathcal{P}).$$

Consider now $E \in \mathcal{E}_N$, $A \in \mathcal{B}_{2M+N}$ with $A \subset E$, and $F \in \bigvee_{n=M+N}^{2M+N+m} f^{-n}\mathcal{P}$, $F \subset G \in \bigvee_{n=M+N}^{2M+N} f^{-n}\mathcal{P}$. Either $A \cap G = \varnothing$ or $A \subset G$. In the second case, we have (by Lemma 3) $f^{M+N}A = f^{M+N}G$ and

$$\frac{\mu(A\cap F)}{\mu(A)}\in\frac{\mu(f^{M+N}F)}{\mu(f^{M+N}G)}[e^{-\beta},e^{\beta}].$$

Now $\mu(f^{M+N}F)=\mu(F)$ and $\mu(f^{M+N}G)=\mu(G)$ since the sets F and G are $\bigvee_{n=M+N}^{\infty}f^{-n}\mathcal{P}$-measurable. Hence

$$\mu(\tilde{E}_N\cap F)=\sum_{A\subset\tilde{E}_N}\mu(A\cap F)=\sum_{A\subset\tilde{E}_N,A\subset G}\frac{\mu(F)}{\mu(G)}\mu(A)(1+\zeta_A)$$

where $|\zeta_A|\leqq e^{\beta}-1$. Thus

$$\left|\mu(\tilde{E}_N\cap F)-\frac{\mu(F)}{\mu(G)}\mu(\tilde{E}_N\cap G)\right|\leqq(e^{\beta}-1)\frac{\mu(F)}{\mu(G)}\mu(\tilde{E}_N\cap G),$$

and so

$$\begin{aligned}
&D\left(\{\tilde{E}_N: E\in\mathcal{E}_N\},\bigvee_{n=M+N}^{2M+N+m}f^{-n}\mathcal{P}\right)\\
&\leqq\sum_{\tilde{E}_N,F}\left\{(e^{\beta}-1)\frac{\mu(F)}{\mu(G)}\mu(\tilde{E}_N\cap G)+\left|\frac{\mu(F)}{\mu(G)}\mu(\tilde{E}_N\cap G)-\mu(\tilde{E}_N)\mu(F)\right|\right\}\\
&\leqq e^{\beta}-1+\sum_{\tilde{E}_N,G}|\mu(\tilde{E}_N\cap G)-\mu(\tilde{E}_N)\mu(G)|\\
&\leqq e^{\beta}-1+2\sqrt{\beta}+D\left(\bigvee_{n=0}^{M}f^{-n}\mathcal{P},\bigvee_{n=M+N}^{2M+N}f^{-n}\mathcal{P}\right).
\end{aligned}$$

Summing up, we have

$$\gamma_{N,m}\leqq 2(\beta+e^{\beta}-1+3\sqrt{\beta})+D(\mathcal{P}_M,f^{-M+N}\mathcal{P}_M)$$

where $\mathcal{P}_M=\bigvee_{n=0}^{M}f^{-n}\mathcal{P}$. Now the fact that (f,μ) is weak-mixing and $\mathcal{P}_M$ is a fixed partition imply [17, p. 41] that one can find a sequence $N_j\to\infty$ with $D(\mathcal{P}_M,f^{-M+N}\mathcal{P}_M)\to 0$ as $j\to\infty$. Given $\varepsilon>0$ by choosing β very small and N appropriately we have $\gamma_{N,m}<\varepsilon$ for all $m\geqq 0$. Thus $\mathcal{P}$ is weak-Bernoulli. □

Example 1. For $\beta>1$ the β-transform $fx=\beta x \pmod 1$ is Bernoulli by Theorems 2(a) and 1. Let U be an open interval. Then length f^nU grows by a factor of β until finally $f^nU\supset(0,a)$ for some $a>0$. Then $f^{m+n}U\supset(0,\beta^m a)$ until one has $f^{m+n}U\supset(0,1)$.

Example 2. Let f_a be linear on each of $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ with $f_a(0)=0$, $f_a(\frac{1}{2})=1$, $f_a(1)=a$. This example was proposed by Ulam [16] and shown to be ergodic by Li and Yorke [7] when $a<\frac{1}{2}$. When $a<1-\sqrt{2}/2$, Theorems 2(c) and 1 show f_a is Bernoulli. For $a=.4$ one checks that f_a maps each of the intervals [.4, .64] and

$[.8, 1]$ into the other one, and that the remainder of $[0, 1]$ contains only two nonwandering points (each fixed). Thus $F(x) = 1$ for $x \in [.4, .64]$, -1 for $x \in [.8, 1]$ and 0 otherwise defines a μ-eigenfunction with $\tau = -1$. So $f_{.4}$ is not weak-mixing.

Added in proof. Under considerably relaxed conditions on f near the a_i's, S. Wong can prove the existence of an invariant measure and M. Ratner the Bernoulli property.

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